

AA 601N : Astrophysical Fluids and Plasma

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References

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- An Introduction to Astrophysical Fluid Dynamics by *Michael J. Thompson*
- Principles of Astrophysical Fluid Dynamics by *Bob Carswell & Cathie Clarke*
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Review of Statistical Mechanics

- Concept of Phase Space
- Concept of *Ensemble* and *Liouville* Theorem
- One and many particle distribution function : BBGKY Hierarchy
- Introducing Equations in Phase space :
 - 1 Collision-less Boltzmann Equation
 - 2 Vlasov Equation.
 - 3 Collision Terms : Fokker Planck, Boltzmann Model
- Concept of Fluid and its description.

Phase Space

Consider a system of N particles. The time evolution of such a system is governed by the *Hamilton's* equation for a given initial conditions.

The Hamiltonian H is a function of canonically conjugate variables : the generalized co-ordinates $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$, corresponding momenta $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N$ and time t .

Hamilton's Equation are -

$$\frac{d\mathbf{q}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i} ; \frac{d\mathbf{p}_i}{dt} = - \frac{\partial H}{\partial \mathbf{q}_i}$$

Therefore, at any given time the system is completely defined if the Hamilton H and initial conditions are known.

Mechanical state of the system \rightarrow single point in a $2N$ dimensional space.

Evolution of that single point \rightarrow $2N$ vectors equations given above.

Such a $2N$ dimensional space made up of N generalized co-ordinates $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ and N momenta $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N$ is called the *Phase space*.

Function in Phase Space

Consider a function $f(\mathbf{q}, \mathbf{p}, t)$ of the $2N$ variables defined in phase space, then its derivative is given by

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^N \frac{\partial f}{\partial \mathbf{q}_i} \cdot \frac{d\mathbf{q}_i}{dt} + \sum_{i=1}^N \frac{\partial f}{\partial \mathbf{p}_i} \cdot \frac{d\mathbf{p}_i}{dt} \\ &= \frac{\partial f}{\partial t} + [f, H]\end{aligned}$$

where $[f, H]$ is called the Poisson bracket and its value is 0 if f is a constant of motion.

Exercise : For a system where Hamilton has no explicit time dependence, prove that the total energy of the system is conserved. What happens when the Hamilton does not have explicit dependence on say \mathbf{q}_k ?

Gibb's Ensemble

A collection of *identical* systems that represent the same average properties is called an Gibb's Ensemble.

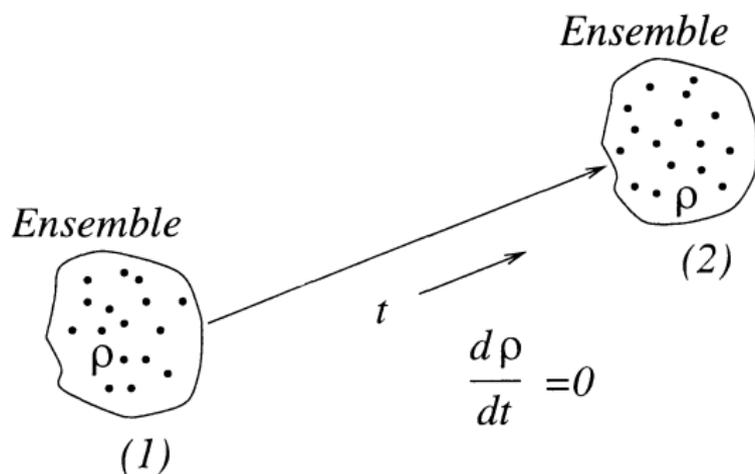
For example, in a harmonic oscillator the total energy is given as $p^2 + q^2$, and this we know remains invariant throughout the motion. Thus each pair (p, q) that preserves this in-variance is a member and a collection of such members forms an Ensemble.

Define density as the number of members in a volume $d\mathbf{q}_1 \dots d\mathbf{q}_N d\mathbf{p}_1 \dots d\mathbf{p}_N$ of the phase space at a given instant of time t .

$$\rho(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N, t) d\mathbf{q}_1 \dots d\mathbf{q}_N d\mathbf{p}_1 \dots d\mathbf{p}_N \quad (1)$$

Liouville Theorem

Liouville Theorem : *The density of states in an ensemble of many identical states with different initial conditions is constant along every trajectory in phase space*



What the Liouville Theorem does not mean ?

- Liouville's theorem does not imply that the density is uniform throughout phase space. In particular, if the Hamiltonian preserves energy, then one trajectory cannot visit two parts of phase space with different energy.
- Liouville's theorem does not imply that every point along a given path has the same density. In other words, suppose that two particles, A and B, follow the same trajectory, except that particle A leads particle B by a finite time (or equivalently, there is a finite distance in xp space between the two particles). Particle A could be in a region of different density than particle B.
- Liouville's theorem only holds in the limit that the particles are infinitely close together. Equivalently, Liouville's theorem does not hold for any ensemble that consists of a finite number of particles; instead the theorem describes the probability density in phase space of an ensemble consisting of an infinite number of possible states.

Liouville Equation

This equation governs the time evolution of density ρ in phase space -

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + [\rho, H]. \quad (2)$$

From the the Liouville's theorem we have L.H.S. = 0

$$\frac{\partial\rho}{\partial t} = -[\rho, H]. \quad (3)$$

Rather than solving set of canonical equations, we can determine the trajectory of system through just the above equation.

This equation also follows super-position principles : **Prove**

Distribution Function

- The probability of finding the system in a volume element $d\mathbf{q}_1 \dots d\mathbf{q}_N d\mathbf{p}_1 \dots d\mathbf{p}_N$ is given by the function -

$$\rho(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N, t) d\mathbf{q}_1 \dots d\mathbf{q}_N d\mathbf{p}_1 \dots d\mathbf{p}_N \quad (4)$$

The specific functional form of ρ in terms of constants of motion is known as the Distribution function.

- According to Liouville's theorem, we will have for a distribution of N particles -

$$\frac{d}{dt} f_N(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_N, t) = 0 \quad (5)$$

- One particle distribution function - $f_1(\mathbf{q}_1, \mathbf{V}_1, t) d\mathbf{q}_1 d\mathbf{V}_1$ is the probability of finding one particle in a volume element $d\mathbf{q}_1 d\mathbf{V}_1$ of the phase space and is obtained by integration of f_N over all other co-ordinates ($\mathbf{q}_2, \mathbf{q}_3, \dots, \mathbf{q}_N, \mathbf{V}_2, \mathbf{V}_3, \dots, \mathbf{V}_N$)
- Similarly, a two particle distribution function - $f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{V}_1, \mathbf{V}_2, t)$ can be defined. This represents joint probability of finding the two particles.

Collision-less Boltzmann Equation

The collision-less Boltzmann Equation can be derived using the Liouville's theorem and it can be shown that -

$$\frac{\partial f_1}{\partial t} + \mathbf{V}_1 \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} + \frac{\mathbf{F}_{\text{ext}}}{m} \frac{\partial f_1}{\partial \mathbf{V}_1} = 0 \quad (6)$$

Several applications of this equation in Astrophysics -

- Study of Stellar motion in the Galaxy or star cluster. :
 $\mathbf{F}_{\text{ext}} = -m\nabla\phi_g$ Poisson Equation : $\nabla^2\phi_g = 4\pi G\rho_m(\mathbf{r}, t)$
where $\rho_m(\mathbf{r}, t) = m \int f(\mathbf{r}, \mathbf{V}, t)d\mathbf{V}$
- Collision-less plasma : The one between Sun and Earth – Space Weather The external force includes an additional term from the Lorentz Force. The Collision-less Boltzmann Equation for plasma in presence of electric field \mathbf{E} and magnetic field \mathbf{B} is called the Vlasov Equation.

Collision Term

In general, the Liouville's Equation for distribution function also has Collisional Terms due to presence of other particles/stars etc. especially in regions of high density.

- **Krook Collision Model** - $\text{RHS} = -\frac{1}{\tau}(f_1 - f_{\text{eq}})$
Question - Can you solve for f_1 assuming no spatial gradients and no-external force?
- **Boltzmann Collision Model** - Restricts the interaction among particles to only binary collisions. Applicable when - a) Particle density is low so that higher order interactions can be neglected. b) Particles experience only short range forces c) Within the range of forces, the short range force dominates over any external force d) The interactions are independent.

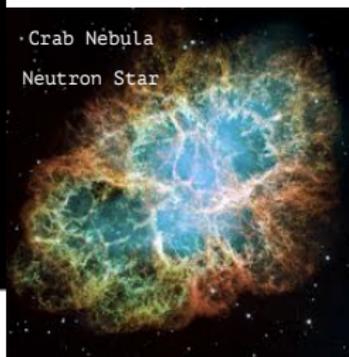
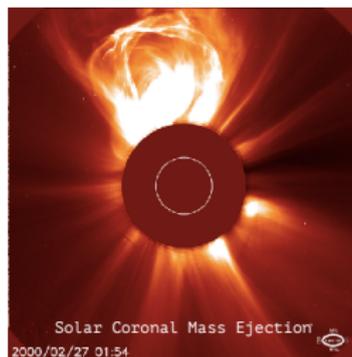
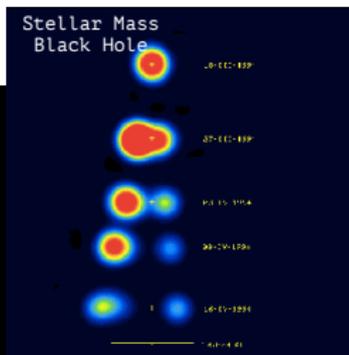
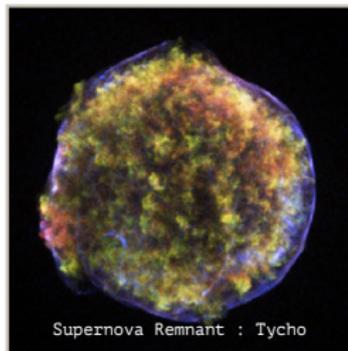
Astrophysical Fluids Dynamics : Area of Application.

Astrophysical fluid dynamics (AFD) is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology itself.

A fluid description is not applicable -

- in regions that are solidified, such as the rocky or icy cores of giant planets (under certain conditions)
- the crusts of neutron stars
- in very tenuous regions where the medium is not sufficiently collisional.

Astrophysical Flows



Astrophysical Fluids Dynamics : Area of Application.

Important Areas of applications include -

- Instabilities in astrophysical fluids
- Convection in stars
- Differential rotation and meridional flows in stars
- Stellar oscillations
- Astrophysical dynamos
- Magnetospheres of stars, planets and black holes
- Interacting binary stars and Roche-lobe overflow
- Tidal disruption and stellar collisions
- Supernovae
- Planetary Nebulae
- Jets and winds from stars and discs
- Star formation and the physics of the interstellar medium
- Astrophysical discs
- Other accretion flows (Bondi, Bondi–Hoyle, etc.)
- Processes related to planet formation and planet–disc interactions
- Planetary atmospheric dynamics
- Galaxy clusters and the physics of the intergalactic medium
- Cosmology and structure formation

Flavors of Astrophysical Fluid Dynamics

- *Basic Model* - A **compressible**, **inviscid** fluid in **Newtonian** (*non-relativistic*) framework → **Hydrodynamics**
- *Thermodynamics* - Treating fluids as either **isothermal**, **adiabatic** or including **radiative process** in various levels of details → **Equation of state**
- *Magnetic fields* - Including the dynamical effects of magnetic fields assuming infinite conductivity → **Ideal Magneto-hydrodynamics**
- *Dissipation Effects* - Include non-ideal effects due to viscosity, magnetic resistivity, Hall effect, Ambipolar diffusion etc.
- *Relativity* - Extend the basic model to incorporate effects due to special and general relativity.

Flavors of Astrophysical Fluid Dynamics

- **HD** - Hydrodynamics (Ideal and Non-Ideal)
- **RaHD** - Radiation Hydrodynamics (*RHD*)
- **SRHD** - Special Relativistic Hydrodynamics (*RHD*)
- **GRHD** - General Relativistic Hydrodynamics

- **MHD** - Magneto-Hydrodynamics (Ideal and Non-Ideal)
- **RaMHD** - Radiation Magneto-Hydrodynamics (*RMHD*)
- **SRMHD** - Special Relativistic Magneto-Hydrodynamics (*RMHD*)
- **GRMHD** - General Relativistic Magneto-Hydrodynamics

- **GRRaMHD** - General Relativistic Radiation Magneto-hydrodynamics.

Concept of Fluid Element

Small size ...

The size of the fluid element, l_{fe} , should be smaller than a scale length for change of any relevant fluid variable q -

$$l_{fe} \ll \frac{q}{|\nabla q|} \quad (7)$$

... yet large enough ...

But at the same time it should be large enough to contain a sufficient number of particles so as to ignore noise due to finite number of particles (*discreteness noise*). Thus for a system with n as the number of particles per unit volume, we should have

$$nl_{fe}^3 \gg 1 \quad (8)$$

... to be *collisional*!

The size of fluid element should be large enough so that the constituent particles *know* about local conditions through collisions -

$$l_{fe} \gg \lambda = \frac{1}{n\sigma} \quad (9)$$

Validity of Fluid Approach

Collisions and Fluid Approach

The equations that govern the dynamics of fluids are essentially derived from micro-physical considerations. The essential idea is that if particles inside a fluid element interact with each other (not necessarily via physical collisions), then they will attain a distribution of particle speed that maximizes the entropy of the system at that temperature. This allows us to define fluid quantities like density, pressure and derive a relation between them in form of Equation of state.

In some cases, in spite of frequent collisions (i.e., $t_{\text{coll}} \ll T_{\text{scale}}$), small deviations to the distribution function of particles can arise. These small deviations can be well accounted for by including appropriate non-ideal effects like viscosity, heat conduction, resistivity etc.

Fluid Approach Fails

Cases where the mean flight time of microscopic particles, $\langle \tau \rangle$ is comparable to characteristic time scale i.e., T_{scale} , the fluid approach is no longer valid. Alternatively, in astrophysical systems where the *mean free path*, $\lambda = \frac{1}{n\sigma}$ is comparable to characteristic length scale, L_{scale} of the system, the fluid equations can not be applied.

Validity of Fluid Approach : Exercise

Astrophysical System	ρ, n	T	L_{scale}	λ^\dagger
Core of Sun-like star	10^2 g cm^{-3}	10^7 K	$\approx 0.05 R_\odot$	$2 \times 10^{-8} \text{ cm}$
Solar Corona	$10^{-15} \text{ g cm}^{-3}$	10^6 K	$\sim 10 Mm$?
ISM-Molecular clouds	10^3 cm^{-3}	10 K	80 pc	?
ISM-Ionized Medium	10^{-3} cm^{-3}	10^6 K	1000-3000 pc	$\sim 3 \text{ pc}$

\dagger NOTE : The *Columb cross section* for collisions, $\sigma \approx 10^{-4} (T / K)^{-2} \text{ cm}^2$ and mean free path $\lambda = \frac{1}{n\sigma}$.

Multi-fluid, Hybrid, Kinetic Approach

In cases where the basic **single-fluid** approach fails, we can adopt more complicated *multi-fluid* or hybrid models which allows us to treat constituent particles separately. For example, in solar corona we can treat ions and electrons separately and study their dynamics along with interactions among them. The most consistent approach is the Kinetic approach, which really solves the *Boltzmann Equation* from first principle, however they can not be applied to study very large systems.

Fluid Variables and Derivatives

Symbols and Meanings

Cartesian co-ordinate

$\mathbf{x} = x\hat{i} + y\hat{j} + z\hat{k}$ and time t .

Fluid Variable	Symbol
Velocity	$\mathbf{v}(\mathbf{x}, t)$
Density	$\rho(\mathbf{x}, t)$
Pressure	$P(\mathbf{x}, t)$
Magnetic Fields	$\mathbf{B}(\mathbf{x}, t)$
Specific Volume	$1/\rho$
Temperature	$\propto P/\rho$
Current Density	$\nabla \times \mathbf{B}$

Lagrangian v/s Eulerian

Eulerian viewpoint - Consider the variation of properties of the fluid at a fixed point in space. (i.e., attached to the inertial co-ordinate system), time derivative and any quantity Q is given by -

$$\frac{\partial Q}{\partial t}$$

Lagrangian viewpoint - Consider the variation of properties of the fluid at a point that moves with the fluid at velocity $\mathbf{v}(\mathbf{x}, t)$, Lagrangian time derivative of quantity Q is given by -

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q$$

More Notations and Vector Calculus

For any variable denoted by $Q(\mathbf{x}, t) \equiv Q(x, y, z, t)$, its partial derivatives are written as -

$$Q_t \equiv \frac{\partial Q}{\partial t}, Q_x \equiv \frac{\partial Q}{\partial x}, Q_y \equiv \frac{\partial Q}{\partial y}, Q_z \equiv \frac{\partial Q}{\partial z}$$

The *dot product* of two vectors $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ is given by -

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Given a scalar quantity ϕ that depends on spatial co-ordinates x , y and z , the gradient operator ∇ as applied to scalar ϕ is a vector given by -

$$\text{grad}\phi \equiv \nabla\phi \equiv (\phi_x, \phi_y, \phi_z) \equiv \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$$

The *divergence operator* applies to any vector \mathbf{A} results in a scalar quantity -

$$\text{div}\mathbf{A} \equiv \nabla \cdot \mathbf{A} \equiv \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$